

ON COLORINGS OF VARIABLE WORDS

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ABSTRACT. In this note, we prove that the base case of the Graham–Rothschild Theorem, i.e., the one that considers colorings of the (1-dimensional) variable words, admits bounds in the class \mathcal{E}^5 of Grzegorczyk’s hierarchy.

1. INTRODUCTION

The Graham–Rothschild Theorem [4] is a generalization of the well know Hales–Jewett Theorem that considers colorings of m -parameter sets instead of constant words. The best known bounds for the Graham–Rothschild Theorem are due to S. Shelah [7] and belong to the class \mathcal{E}^6 of Grzegorczyk’s hierarchy. In this note we consider the “base” case of the Graham–Rothschild Theorem, that concerns colorings of (1-dimensional) variable words. We obtain bounds for this base case in \mathcal{E}^5 of Grzegorczyk’s hierarchy. Although the proof is an appropriate modification of S. Shelah’s proof for the Hales–Jewett Theorem, it is streamline and independent.

The base case of the Graham–Rothschild Theorem is of particular interest, since it is the one needed for the proof of the density Hales–Jewett Theorem in [2]. Moreover, it has as an immediate consequence the finite version of the Carlson–Simpson Theorem on the left variable words and therefore the finite version of the Halpern–Läuchli theorem for level products of homogeneous trees (see also [8]).

To state the result of this note, we need some pieces of notation. Let k and n be positive integers. By $[k]$ we denote the set $\{1, \dots, k\}$ and $[k]^n$ the set of all sequences (a_0, \dots, a_{n-1}) of length n taking values in $[k]$. We view $[k]$ as a finite alphabet and the elements of $[k]^n$ as words. Thus, by the term *word over k of length n* we mean an element of $[k]^n$. Also let m be a positive integer and v, v_0, \dots, v_{m-1} distinct symbols not belonging to $[k]$. We view these symbols as variables. A variable word $w(v)$ over k is a sequence in $[k] \cup \{v\}$, where the variable v occurs at least once. More generally, an m -dimensional variable word $w(v_0, \dots, v_{m-1})$ over k is a sequence in $[k] \cup \{v_0, \dots, v_{m-1}\}$ such that each v_j occurs at least once and they are in block position, meaning that if $w(v_0, \dots, v_{m-1})$ is of the form (x_0, \dots, x_{n-1}) then $\max\{i : x_i = v_j\} < \min\{i : x_i = v_{j+1}\}$ for all $0 \leq j < m-1$. Clearly, every variable word can be viewed as an 1-dimensional variable word.

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Let k, m be positive integers and $w(v_0, \dots, v_{m-1})$ an m -dimensional variable word over k . For every sequence of symbols $\mathbf{x} = (x_i)_{i=0}^{m-1}$ of length m we denote by $w(\mathbf{x})$ the sequence resulting by substituting each occurrence of v_i by x_i for all $0 \leq i < m$. Observe that $w(\mathbf{x})$ is an m' -dimensional variable word, for some $m' \leq m$, if and only if \mathbf{x} is an m' -dimensional variable word. In particular, $w(\mathbf{x})$ is a variable word if and only if \mathbf{x} is a variable word. An m' -dimensional (resp. single) variable word is called reduced by $w(v_0, \dots, v_{m-1})$ if it is of the form $w(\mathbf{x})$ for some m -dimensional (resp. single) variable word \mathbf{x} of length m .

Theorem 1. *For every triple of positive integers k, r, m there exists a positive integer n_0 with the following property. For every integer n with $n \geq n_0$ and every r -coloring of all the variable words over k of length n , there exists an m -dimensional variable word $w(v_0, \dots, v_{m-1})$ over k of length n such that the set of all variable words over k reduced by $w(v_0, \dots, v_{m-1})$ is monochromatic. We denote the least such n_0 by $GR(k, m, r)$.*

Moreover, the numbers $GR(k, m, r)$ are upper bounded by a primitive recursive function belonging to the class \mathcal{E}^5 of Grzegorzczuk's hierarchy.

2. THE HINDMAN THEOREM

The case “ $k = 1$ ” of Theorem 1 follows by the finite version of Hindman's theorem [6]. To state it we need some pieces of notation.

Let n, m, d be positive integers with $d \leq m \leq n$. We denote by $\mathcal{F}(n)$ the set of all non-empty subsets of $\{0, \dots, n-1\}$. A finite sequence $\mathbf{s} = (s_i)_{i=0}^{m-1}$ in $\mathcal{F}(n)$ is called block if $\max s_i < \min s_{i+1}$ for all $0 \leq i < m-1$. We denote the set of all block sequences of length m in $\mathcal{F}(n)$ by $\text{Block}^m(n)$. For every $\mathbf{s} = (s_i)_{i=0}^{m-1}$ in $\text{Block}^m(n)$ we define the set of nonempty unions of \mathbf{s} to be

$$\text{NU}(\mathbf{s}) = \left\{ \bigcup_{i \in t} s_i : t \text{ is a nonempty subset of } \{0, \dots, m-1\} \right\}.$$

We say that a block sequence $\mathbf{t} = (t_i)_{i=0}^{d-1}$ in $\mathcal{F}(n)$ is a block subsequence of \mathbf{s} if $t_i \in \text{NU}(\mathbf{s})$ for all $0 \leq i < d$. The finite version of Hindman's theorem is stated as follows.

Theorem 2. *For every pair m, r of positive integers, there exists a positive integer n_0 with the following property. For every finite block sequence \mathbf{s} of nonempty finite subsets of \mathbb{N} of length at least n_0 and every coloring of the set $\text{NU}(\mathbf{s})$ with r colors, there exists a block subsequence \mathbf{t} of \mathbf{s} of length m such that the set $\text{NU}(\mathbf{t})$ is monochromatic. We denote the least n_0 satisfying the above property by $H(m, r)$.*

Moreover, the numbers $H(m, r)$ are upper bounded by a primitive recursive function belonging to the class \mathcal{E}^4 of Grzegorzczuk's hierarchy.

This finite version follows by the disjoint union theorem [4, 9] and Ramsey's theorem. The bounds for the disjoint union theorem given in [9], as well as, the

bound for the Ramsey numbers given in [3] are in \mathcal{E}^4 . Using these bounds, one can see that the numbers $H(m, r)$ are upper bounded by a primitive recursive function belonging to the class \mathcal{E}^4 of Grzegorzczuk's hierarchy. We refer the interested reader to [1].

3. INSENSITIVITY

The proof of Theorem 1 proceeds by induction on k . The main notion that helps us to carry out the inductive step of the proof is an appropriate modification of Shelah's insensitivity (see Definition 3 below).

First, let us introduce some additional notation. Let k, m, n be positive integers with $m \leq n$ and $w(v_0, \dots, v_{m-1})$ be an m -dimensional variable word over k of length n . We denote by $W_v^k(n)$ the set of all variable words over k of length n , while by $W_v^k(w(v_0, \dots, v_{m-1}))$ the set of all variable words over k reduced by $w(v_0, \dots, v_{m-1})$. If $w = w(v_0, \dots, v_{m-1}) = (x_i)_{i=0}^{n-1}$, for every $j = 0, \dots, m-1$ we set

$$\text{supp}_w(v_j) = \{i \in \{0, \dots, n-1\} : x_i = v_j\}.$$

We consider the following analogue of Shelah's insensitivity.

Definition 3. Let k, m, n be positive integers with $m \leq n$. Also let $w(v_0, \dots, v_{m-1})$ be an m -dimensional variable word over $k+1$ of length n and a, b in $[k+1]$ with $a \neq b$.

- (i) We say that two words $\mathbf{x} = (x_i)_{i=0}^{n-1}$ and $\mathbf{y} = (y_i)_{i=0}^{n-1}$ over $k+1$ of length n are (a, b) -equivalent if for every e in $[k+1] \setminus \{a, b\}$, we have that $x_i = e$ if and only if $y_i = e$ for all i in $\{0, \dots, n-1\}$.
- (ii) We say that a coloring c of $W_v^{k+1}(n)$ is (a, b) -insensitive over $w(v_0, \dots, v_{m-1})$ if for every pair \mathbf{x}, \mathbf{y} of (a, b) -equivalent words over $k+1$ of length m , we have that $c(w(\mathbf{x})) = c(w(\mathbf{y}))$.

We prove the following analogue of Shelah's insensitivity lemma.

Lemma 4. For every triple k, m, r of positive integers there exists a positive integer n_0 satisfying the following. For every integer n with $n \geq n_0$, every a, b in $[k+1]$ with $a \neq b$ and every r -coloring c of $W_v^{k+1}(n)$ there exists an m -dimensional variable word $w(v_0, \dots, v_{m-1})$ over $k+1$ of length n such that c is (a, b) -insensitive over $w(v_0, \dots, v_{m-1})$. We denote the least such n_0 by $\text{Sh}_v(k, m, r)$.

Finally, the numbers $\text{Sh}_v(k, m, r)$ are upper bounded by a primitive recursive function belonging to the class \mathcal{E}^4 of Grzegorzczuk's hierarchy.

Before we proceed to the proof of Lemma 4 let us define a function $f : \mathbb{N}^4 \rightarrow \mathbb{N}$ by the following rule. For every choice of positive integers k, m, r we recursively define

$$\begin{cases} f(k, 0, m, r) = 0, \\ f(k, j+1, m, r) = f(k, j, m, r) + r^{(k+2)^{m-j-1} + f(k, j, m, r)} \end{cases}$$

and we set $f(k, j, m, r) = 0$ if at least one of the integers k, m, r is equal to zero. Observe that f belongs to the class \mathcal{E}^4 of Grzegorzczuk's hierarchy.

Proof of Lemma 4. Let k, m, r of positive integers. We will show the inequality

$$(1) \quad \text{Sh}_v(k, m, r) \leq f(k, m, m, r).$$

Indeed, let n be an integer with $n \geq f(k, m, m, r)$ and $c : W_v(n) \rightarrow \{1, \dots, r\}$. Also let a, b in $[k+1]$ with $a \neq b$. Set

$$q_j = f(k, m - j, m, r) + j$$

for all $0 \leq j \leq m$. We inductively construct a sequence $(w_j)_{j=0}^m$ satisfying for every $j = 0, \dots, m$ the following.

- (i) w_j is a q_j -dimensional variable word over $k+1$ of length n .
- (ii) If $0 < j$, then w_j is reduced by w_{j-1} .
- (iii) If $1 < j$, then $\text{supp}_{w_j}(v_{j-2}) = \text{supp}_{w_{j-1}}(v_{j-2})$.
- (iv) If $0 < j$, then for every $\mathbf{x} = (x_i)_{i=0}^{q_j-1}, \mathbf{y} = (y_i)_{i=0}^{q_j-1}$ in $W_v^{k+1}(q_j)$ such that
 - (1) $x_i = y_i$ for all $i = 0, \dots, q_j - 1$ with $i \neq j - 1$ and
 - (2) $x_{j-1} = a$ and $y_{j-1} = b$,
 we have that $c(w_j(\mathbf{x})) = c(w_j(\mathbf{y}))$.

We pick an arbitrary q_0 -dimensional variable word $w_0 = w_0(v_0, \dots, v_{q_0-1})$ over $k+1$ of length n . Clearly, condition (i) above is satisfied while conditions (ii)-(iv) are meaningless. Let us assume that for some positive j we have constructed w_0, \dots, w_{j-1} satisfying the conditions above. Set $d = r^{(k+2)^{q_j-1}}$ and observe that

$$(2) \quad q_j + d = q_{j-1} + 1.$$

Moreover, for every $t = 0, \dots, d$ we set

$$\mathbf{a}_t = (\underbrace{a, \dots, a}_{t\text{-times}}, \underbrace{b, \dots, b}_{(d-t)\text{-times}})$$

and $A = \{\mathbf{a}_t : t = 0, \dots, d\}$. We define a map Q from $A \times W_v^{k+1}(q_j - 1)$ into $W_v^{k+1}(q_{j-1})$ setting for each t in $\{0, \dots, d\}$ and $(z_i)_{i=0}^{q_j-2}$ in $W_v^{k+1}(q_j - 1)$

$$Q(\mathbf{a}_t, (z_i)_{i=0}^{q_j-2}) = (z_i)_{i=0}^{j-2} \frown \mathbf{a}_t \frown (z_i)_{i=j-1}^{q_j-2},$$

under the convention that $(z_i)_{i=0}^{j-2}$ (resp. $(z_i)_{i=j-1}^{q_j-2}$) is the empty sequence if $j = 1$ (resp. $j = m$). We denote by \mathcal{X} the set of all maps from $W_v^{k+1}(q_j - 1)$ into $\{1, \dots, r\}$. Clearly, \mathcal{X} is of cardinality at most d . For every t in $\{0, \dots, d\}$, we define T_t in \mathcal{X} setting for every \mathbf{z} in $W_v^{k+1}(q_j - 1)$

$$T_t(\mathbf{z}) = c(w_{j-1}(Q(\mathbf{a}_t, \mathbf{z}))).$$

Since the cardinality of \mathcal{X} is at most d , there exist t_1, t_2 in $\{0, \dots, d\}$ such that $t_1 < t_2$ and $T_{t_1} = T_{t_2}$. Finally, we set

$$w'(v_0, \dots, v_{q_j-1}) = (v_i)_{i=0}^{j-2} \frown (\underbrace{a, \dots, a}_{t_1\text{-times}}, \underbrace{v_{j-1}, \dots, v_{j-1},}_{(t_2-t_1)\text{-times}} \underbrace{b, \dots, b}_{(d-t_2)\text{-times}}) \frown (v_i)_{i=j}^{q_j-1},$$

under the convention that $(v_i)_{i=0}^{j-2}$ (resp. $(v_i)_{i=j}^{q_j-1}$) is the empty sequence if $j = 1$ (resp. $j = m$), and $w_j(v_0, \dots, v_{q_j-1}) = w_{j-1}(w'(v_0, \dots, v_{q_j-1}))$. By equation (2), we have that w' is of length q_{j-1} and therefore w_j is well defined. It is immediate that $w_j(v_0, \dots, v_{q_j-1})$ satisfies conditions (i)-(iii). Let $\mathbf{x} = (x_i)_{i=0}^{q_j-1}, \mathbf{y} = (y_i)_{i=0}^{q_j-1}$ in $W_v^{k+1}(q_j)$ as in condition (iv). Define $\mathbf{z} = (z_i)_{i=0}^{q_j-2}$ setting $z_i = x_i$ if $i < j-1$ and $z_i = x_{i+1}$ otherwise. Observe that

$$Q(\mathbf{a}_{t_2}, \mathbf{z}) = w'(\mathbf{x}) \text{ and } Q(\mathbf{a}_{t_1}, \mathbf{z}) = w'(\mathbf{x}).$$

Therefore,

$$\begin{aligned} c(w_j(\mathbf{x})) &= c(w_{j-1}(w'(\mathbf{x}))) = c(w_{j-1}(Q(\mathbf{a}_{t_2}, \mathbf{z}))) = T_{t_2}(\mathbf{z}) = T_{t_1}(\mathbf{z}) \\ &= c(w_{j-1}(Q(\mathbf{a}_{t_1}, \mathbf{z}))) = c(w_{j-1}(w'(\mathbf{y}))) = c(w_j(\mathbf{y})) \end{aligned}$$

as desired and the proof of the inductive step of the construction is complete.

Let us set $w(v_0, \dots, v_{m-1}) = w_m(v_0, \dots, v_{m-1})$ and observe that $w(v_0, \dots, v_{m-1})$ is as desired. Indeed, first observe that by condition (ii) of the inductive construction we have that w is reduced by w_j for all j in $\{0, \dots, m\}$. Moreover, by condition (ii) we have that $\text{supp}_w(v_{j-1}) = \text{supp}_{w_j}(v_{j-1})$ for all j in $\{1, \dots, m\}$. Thus, for every j in $\{1, \dots, m\}$ and every $\mathbf{x}' = (x'_i)_{i=0}^{m-1}, \mathbf{y}' = (y'_i)_{i=0}^{m-1}$ in $W_v^{k+1}(m)$ such that $x'_{j-1} = a, y'_{j-1} = b$ and $x'_i = y'_i$ for all $i \neq j-1$, there exist $\mathbf{x} = (x_i)_{i=0}^{q_j-1}, \mathbf{y} = (y_i)_{i=0}^{q_j-1}$ in $W_v^{k+1}(q_j)$ satisfying:

- (a) $x_{j-1} = a$ and $y_{j-1} = b$
- (b) $x_i = y_i$ for all $i = 0, \dots, q_j - 1$ with $i \neq j - 1$
- (c) $w(\mathbf{x}') = w_j(\mathbf{x})$ and $w(\mathbf{y}') = w_j(\mathbf{y})$

and therefore, by condition (iv) we have that

$$(3) \quad c(w(\mathbf{x}')) \stackrel{(c)}{=} c(w_j(\mathbf{x})) \stackrel{(iv)}{=} c(w_j(\mathbf{y})) \stackrel{(c)}{=} c(w(\mathbf{y}')).$$

One can easily see that (3) implies that the coloring c in (a, b) -insensitive over $w(v_0, \dots, v_{m-1})$. Thus inequality (1) is valid and since f belongs to the class \mathcal{E}^4 of Grzegorzczuk's hierarchy, the proof of the lemma is complete. \square

4. PROOF OF THEOREM 1

As we mentioned in the introduction, the proof of Theorem 1 is a modification of S. Shelah's proof for the Hales–Jewett Theorem. It proceeds by induction on k . For “ $k = 1$ ” Theorem 1 follows readily by the finite version of Hindman's theorem, that is, Theorem 2. In particular, we have

$$(4) \quad \text{GR}(1, m, r) = \text{H}(m, r).$$

Towards the proof of the inductive step, we, in particular, show the following inequality.

$$(5) \quad \text{GR}(k+1, m, r) \leq \text{Sh}_v(k, \text{GR}(k, m, r), r).$$

Indeed, let us set $M = \text{GR}(k, m, r)$ and pick any integer n with $n \geq \text{Sh}_v(k, M, r)$. Also, let c be an r -coloring of $W_v^{k+1}(n)$. By Lemma 4, there exists an M -dimensional variable word $w'(v_0, \dots, v_{M-1})$ over $k+1$ of length n such that the coloring c is $(k, k+1)$ -insensitive. We define an r -coloring c' on $W_v^k(M)$ by setting

$$c'(\mathbf{x}) = c(w'(\mathbf{x}))$$

for all \mathbf{x} in $W_v^k(M)$. By the definition of M , there exists an m -dimensional variable word $w''(v_0, \dots, v_{m-1})$ over k of length M such that the set $W_v^k(w'')$ is c' -monochromatic. We set $w(v_0, \dots, v_{m-1}) = w'(w''(v_0, \dots, v_{m-1}))$. Clearly, w is reduced by w' and therefore, since c is $(k, k+1)$ -insensitive over w' , we have that c is $(k, k+1)$ -insensitive over w too. Moreover, by the definition of c' and the choice of w'' , we have that the set $\{w(\mathbf{x}) : \mathbf{x} \in W_v^k(m)\}$ is c -monochromatic. Invoking the insensitivity of the coloring c over w we have that $W_v^{k+1}(w)$ is monochromatic as desired.

Finally, by (4),(5) and the fact that both the numbers $H(m, r)$ and $\text{Sh}_v(k, m, r)$ are upper bounded by a primitive recursive function belonging to the class \mathcal{E}^4 of Grzegorczyk's hierarchy, we have that the numbers $\text{GR}(k, m, r)$ are upper bounded by a primitive recursive function belonging to the class \mathcal{E}^5 .

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